Math 565: Functional Analysis
Lecture 16

Examples (continued).

(c) Let X be a lcH space, e.g. $\mathbb{R}^d$, and recall that $C(X)$ is the space of continuous functions on X. The natural topology on this space is the compact-open topology, i.e. the one generated by the sets $[K;U]$, where $K \subseteq X$ is compact and $U \subseteq \mathbb{C}$ open and
$$[K;U] := \{f \in C(X) : f(K) \subseteq U\}.$$
This top. is the same as the top. of uniform convergence on compact sets, i.e. generated by the seminorms p_K, where $K \subseteq X$ is compact, $p_K(f) := \sup_{x \in K} |f(x)|$. When X is σ-compact so $X = \bigcup_{n \in \mathbb{N}} K_n$, then the top. is generated by the seminorms $\{p_{K_n}\}_{n \in \mathbb{N}}$ and hence it is metrizable.

We now prove the analogue of "continuous linear ⇒ Lipschitz" for top. vector spaces generated by seminorms.

Prop. Let X, Y be top. vector spaces whose topologies are generated by families P and Q of semi-norms, resp. Then every linear $T: X \to Y$ is continuous ⇔ for each $q \in Q$ there is $C > 0$ and finite $P_0 \subseteq P$ such that $\forall x \in X$
$$q(Tx) \leq C \sum_{p \in P_0} p(x). \qquad (*)$$

Proof. ⇐. Suffices to show continuity at $0 \in X$ but if right side of $(*) \to 0$, so does the left side.

⇒. Recall that finite intersections of balls in different seminorms from P form a basis for the top. on X. Thus, by continuity at $0 \in X$, $T^{-1}(B_1^q)$ is an open neighborhood of $0 \in X$, hence contains $\bigcap_{p \in P_0} B_\delta^p$ where $P_0 \subseteq P$ is finite and $\delta > 0$. Hence $q(Tx) < 1$ whenever $p(x) < \delta$ for all $p \in P_0$. By scaling, we get: $\sum_{p \in P_0} p(x) < r \cdot \delta \Rightarrow q(Tx) < r$. Thus, $\sum_{p \in P_0} p(x) \leq r \cdot \delta \Rightarrow q(Tx) \leq r$ $(*)$

because if $\sum_{p\in P_0} p(x) \le r\delta$, then $\forall \varepsilon>0$ $\sum_{p\in P_0} p(x) < (r+\varepsilon)\delta \Rightarrow q(T_x) < r+\varepsilon$ hence $q(T_x)\le r$.

For each $x\in X$, we have two cases:

Case 1: $\sum_{p\in P_0} p(x) > 0$. Then $y := \frac{1}{\sum_{p\in P_0} p(x)} x$ satisfies $\sum_{p\in P_0} p(y) = 1$, so by $(*)$, we get

$$q(T_y) \le \frac{1}{\delta}, \text{ hence } q(T_x) \le \frac{1}{\delta}\sum_{p\in P_0} p(x).$$

Case 2: $\sum_{p\in P_0} p(x) = 0$. Then the hypothesis of $(*)$ holds for all $r>0$, hence $q(T_x)\le r$ for all $r>0$, thus $q(T_x) = 0 = \frac{1}{\delta}\sum_{p\in P_0} p(x)$.

Hence in either case $C := \frac{1}{\delta}$ works. $\square$

Cor. Let X be a top. vector space generated by a family P of seminorms. Then X^* is very rich, more precisely, for every finite $P_0\subseteq P$ and $C>0$, there is a linear functional $f: X\to\mathbb{C}$ satisfying, for every $x\in X$,

$$|f(x)| \le C\cdot \sum_{p\in P_0} p(x),$$

so $f\in X^*$.

Example. Let $C^\infty := C^\infty([0,1])$ be the space of all infinitely differentiable functions on $[0,1]$. This space is closed under differentiation operator $D: C^\infty\to C^\infty$ by $f\mapsto f'$, and we would like to equip C^∞ with a topology with respect to which D would be continuous. This topology cannot come from a norm because then continuity would imply boundedness while D is not bounded because $D(e^{nx}) = n\cdot e^{nx}$ so $\|D\|\ge n$ for all $n\in\mathbb{N}$. Instead, we equip C^∞ with the top. generated by the seminorms $p_k(f) := \|f^{(k)}\|_u$. Then this top is metrizable and D is continuous because $p_k(Df) = \|f^{(k+1)}\|_u = p_{k+1}(f)$. Also this top. is "complete" in the following sense.

Def. In a top. v.s. X, a sequence $(x_n) \subseteq X$ is called Cauchy if $(x_n - x_m) \to 0$ as $\min(n,m) \to \infty$. More generally, a net $(x_i)_{i \in I} \subseteq X$ is called Cauchy if $(x_i - x_j) \to 0$ as $(i,j) \to \infty$. We say that X is complete if every Cauchy net converges. If X is 1^{st} ctbl, then X is complete $\Leftrightarrow$ every Cauchy sequence converges.

Remark. If a top. v.s. X is complete, then every compatible translation-invariant metric d on X is complete: if (x_n) is a d-Cauchy sequence, then $d(x_n, x_m) \to 0$ as $\min(n,m) \to \infty$ but $d(x_n, x_m) = d(x_n - x_m, 0)$ by translation invariance, so (x_n) is Cauchy in the sense above, so it converges in X. Also note that if X is metrizable, then by the Birkhoff-Kakutani theorem, it admits a compatible translation-invariant metric.

Def. A top. v.s. X is called Fréchet if it complete and generated by a ctbl family of seminorms which separate points (hence metrizable).

Examples. All examples so far are complete top. vector spaces. HW

(a) $L^1_{loc}(\mathbb{R}^d)$ is complete by the same proof as $L^1(B_n)$ is complete, where $B_n \subseteq \mathbb{R}^d$ is the ball of radius $n \in \mathbb{N}$. Hence $L^1_{loc}(\mathbb{R}^d)$ is a Fréchet space.

(b) We proved in class that $L^0(X, \mu)$ equipped with the convergence in measure top. is complete.

(c) $C(X)$ equipped with the convergence-on-compact-sets top. is complete similarly to (a) but with compact sets instead of balls B_n. $C(X)$ is Fréchet if X is σ-compact.

(d) $C^\infty([0,1])$ is also a Fréchet space because for each seminorm on its own is "complete" like in the other example.

Detour on product topology.

Given a set X and a family of maps $\{T_i\}_{i \in I}$, $T_i : X \to Y_i$, where Y_i is some top. space, the topology on X generated by $(T_i)_{i\in I}$ is the smallest topology which makes all T_i, $i\in I$, continuous; namely, the topology on X generated by $T_i^{-1}(V_i)$ where $i\in I$, $V_i\subseteq Y_i$ open. Note that in this topology, $x_n \to x \iff T_i(x_n)\to T_i(x)$ for all $i\in I$.

Example. Let X be a vector space and Y be a normed vector space. Let $\{T_i\}_{i\in I}$ be a family of linear maps $X\to Y$. Then the top. generated by $\{T_i\}_{i\in I}$ makes X into a convex top. vector space. This top. is the same as the one generated by the seminorms $p_i(x) := \|T_i(x)\|$.

Subexample. The top. on $C^\infty([0,1])$ given by the seminorms $p_k(f) := \|f^{(k)}\|_u$ is such an example, where $(T_k)_{k\in\mathbb{N}}$ are the linear maps $T_k f := f^{(k)}$ and $Y := C([0,1])$ with the uniform norm $\|\cdot\|_u$.

Example (product top). Let $(X_i)_{i\in I}$ be a family of top spaces and denote

$$\prod_{i\in I} X_i := \{ f : I \to \bigcup_{i\in I} X_i \;:\; f(i)\in X_i \text{ for all } i\in I\}.$$

The product top on $X := \prod_{i\in I} X_i$, also called the topology of pointwise convergence is generated by the projection/evaluation maps $\pi_i : X\to X_i : x\mapsto x(i)$.
Thus, $x_n\to x$ in the product top $\iff x_n(i)\to x(i)$ for all $i\in I$.
Tychonoff's theorem states that products of compact spaces are compact.